A Simple Explicit Model Approximating the Relationship between Speed and Density of Vehicular Traffic on Urban Roads

Abhik Kumar Das and Jai Asundi

***Abstract* — With the increase in simulation of urban environments for the purpose of planning, modelling vehicular traffic has become important. While empirical evidence on traffic flow is relatively sparse, models representing the same are being increasingly used for planning urban roads and environments. In this paper, a simple explicit model is proposed to approximate the speed versus density of vehicular traffic flow. The model, which uses two parameters, derived from simple measurement on real time traffic data, allows for a prediction of the approximate relationship for congested as well as uncongested vehicular traffic flow. The proposed model is especially useful in conditions where available data is sparse and can be invaluable for the modelling and simulation of urban environments.**

I. Introduction

MODELLING vehicular traffic for the purpose of planning of roads and/or other urban infrastructure has become pervasive in recent times[1-4]. As a result, it is very important that the models are used adequately and accurately represent the characteristics of the roads being modelled. However, empirical data on traffic flow is relatively sparse. Collection of real-world data is time-consuming, expensive, and cumbersome. It is thus important that the models being used to estimate traffic flow are not data intensive, yet sufficiently represent the phenomena of interest.

In this paper, we propose a simple explicit model that relates the vehicular speed to the vehicular density on a given road segment. Our proposed model is of a general form and the parameters can be estimated with a very sparse set of data. We demonstrate how this model can be used with an existing real-world dataset and how it is reasonable in its approximation.

Our paper is structured as follows: Section II describes the previous work in this area. In Section III, we describe our model, and Section IV compares our model to the existing models. Section V describes how the parameter values are to be estimated from sample data. In Section VI, we use some real-world data to explain our model and how it can be used. We conclude with some remarks and caveats about our proposed model.

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II. Previous Work

A generic traffic flow model can be represented as

$$\partial_t \rho + \partial_x f(\rho) = 0, \quad x \in \mathbb{R}, t > 0$$
$$\rho(x,0) = \rho_0(x), \qquad x \in \mathbb{R}$$

where ρ is the traffic density and $f(\rho)$ can be viewed as the outward flux of the scalar density function ρ. The 'flow' parameter of traffic model, which can be defined as the number of vehicles passed at fixed point in unit time, can be represented as the outward flux $f(\rho)$. Due to Edie [5], for stationary traffic, the flow parameter, $f(\rho)$ equals space-mean speed times density i.e. $f(\rho) = \rho v$, where v is the space mean speed. Traffic researchers have long been interested in functionally specifying and estimating these relations [6, 7]. Greenshield's [6] data suggested a linear speed-density relation, leading him to propose a parabolic function as an approximation to the flow-density relation. Other functional forms, based on notions like fluid dynamics and car following decisions, give rise to a variety of forms. Considering the free-flow speed (which can be considered as the maximum speed) as v_f, the maximum density ρ_{max}, Lighthill-Whitham-Richards linear model [8] can be represented as

$$v(\rho) = v_f\left(1 - \frac{\rho}{\rho_{max}}\right) \qquad \text{for } 0 \le \rho \le \rho_{max} \tag{1}$$

Greenberg [9] proposed a logarithmic form for speed versus density,

$$v(\rho) = v_f \ln\frac{\rho_{max}}{\rho} \qquad \text{for } 0 < \rho \le \rho_{max} \tag{2}$$

We can consider that at any instant the traffic density (ρ_0) produces the traffic speed as v_0. Then the Greenshield linear model [6] can be expressed as

$$v(\rho) = v_0\left(2 - \frac{\rho}{\rho_0}\right) \qquad \text{for } 0 < \rho \le \rho_{max} \tag{3}$$

Underwood [10] used an exponential form,

$$v(\rho) = v_0 \exp\left(1 - \frac{\rho}{\rho_0}\right) \qquad \text{for } 0 < \rho \le \rho_{max} \tag{4}$$

Greenberg [9] used the logarithmic form also as

$$v(\rho) = v_0\left(1 - \ln\frac{\rho}{\rho_0}\right) \qquad \text{for } 0 < \rho \le \rho_{max} \qquad (5)$$

Drake et al. [7] proposed the model resembling Underwood [10] as

$$v(\rho) = v_0 \exp\left\{\frac{1}{2}\left(1 - \frac{\rho}{\rho_0}\right)^2\right\} \text{for } 0 < \rho \le \rho_{max} \qquad (6)$$

The 'stimulus-response' car-following models of Gazis, Herman, and Rothery [11] have a general form in which the acceleration of a following vehicle responds to the separation and difference in speed from the vehicle in front but this model is complex due to the presence of space derivatives. Papacostas [12] proposed a model expressible as an extension of Duncan's model [13, 14], and Newell's model [15], somewhat rearranged version of Underwood's model [10] with the additional equation on hypothetical speed at zero density. Random driver behaviors are analyzed in [16] to investigate traffic variability, and Kockelman [17] uses the interacting information on travellers, weather, and vehicle type with density for a least-squares polynomial model of flow. Laval and Leclercq [18] describe their model through a merge where traffic reached unsustainable flow and density values, which both gradually 'relaxed' to lower values, in an almost orthogonal direction to the steady-state flow-density relationship. Some researchers also used microscopic modelling of the relaxation phenomenon using a macroscopic lane-changing model [18]. Addison and Heydecker [19] uses different traffic speed limits in their model which found that jam density is not a property of moving traffic.

In this paper we propose a new model to approximate the speed versus density relationship of traffic flow. The functional relationship between the speed and traffic density is simple and explicit and shows the average trend of these two basic traffic variables.

III. Approximate Explicit Model for Speed-Density Curve

The proposed model for the speed versus density of the traffic flow can be represented as

$$\frac{v}{v_{max}} = 1 - (1 - m)\frac{\rho}{\rho_{max}} - m\left(\frac{\rho}{\rho_{max}}\right)^n \qquad (7)$$

for $0 \le \rho \le \rho_{max}$, where m and n are two vehicle related parameters depending on the statistical behavior of the traffic. Here when $\rho \to \rho_{max}$, $v \to 0$ and for $\rho \to 0$, $v \to v_{max}$. When ρ/ρ_{max} is small, the normalized speed v/v_{max} can be viewed as linear with normalized density ρ/ρ_{max} and the 'linear factor', m captures the linear dependency of speed with density mostly in uncongested traffic flow. When the density is over a critical point like a congested traffic, the speed drastically reduces with density and tends to zero for maximum density, a condition which may arise at the time of traffic jam when the average traffic speed is very low and traffic density is comparatively high. A power law term can be used in the proposed model to incorporate the effect of congested traffic situation when the normalized density ρ/ρ_{max} is high and tends to unity. The term n used in the proposed model (7) can be regarded as the 'power factor' of the speed-density relationship of the traffic model.

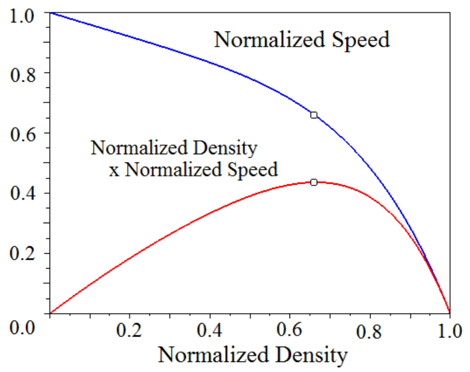


Fig 1: represents the normalized speed vs normalized density and the flow vs normalized density using the proposed power law model using *m*=0.6 ans *n* =5. The marked point represents the critical point where flow is at maximum.

As shown in Fig 1(a), normalized speed decreases almost linearly with normalized density up to a critical point where the product of speed and density (which is defined as traffic flow earlier) is maximum and while the normalized density is above this critical point, the normalized speed reduces very rapidly. The traffic flow versus normalized speed is shown in Fig 1(b) which shows that there is a critical point (critical density) for which the flow is maximum.

IV. Relation of Proposed Model with Other Basic Speed-Density Model

In this section, we show that how the proposed model is generic and how the other basic simple explicit speed-density functional form of traffic flow (as discussed earlier) can be generated from the proposed model. For the simplification, we have considered the most basic models of speed-density curve like linear, logarithmic, and exponential models.

A. *Linear Model*

To normalize the linear model (3), we can use the fact that $\rho \to \rho_{max}$ implies $v \to 0$, hence $\rho_{max} = 2\rho_0$. Similarly for $\rho \to 0$, $v \to v_{max}$, hence $v_{max} = 2v_0$. Now (3) can be normalized as

$$\frac{v}{v_{max}} = 1 - \frac{\rho}{\rho_{max}} \qquad (8)$$

Equation (8) is same as the LWR model (1). Now if we put m = 0 in the (7) we get (8). Hence both Equations (1) and (3) can be generated from the proposed model (7).

B. *Logarithmic Model*

In (5), we can use the similar analogy that $\rho \to \rho_{max}$ implies $v \to 0$, hence $\rho_{max} = e\rho_0$. Now Equation (4) can be converted into

$$\frac{v}{v_{max}} = \frac{v_0}{v_{max}}\left(1 - \ln\frac{\rho}{\rho_{max}}\frac{\rho_{max}}{\rho_0}\right)$$
$$= \frac{v_0}{v_{max}}\left(1 - \ln\frac{\rho}{\rho_{max}} - \ln e\right) = \frac{v_0}{v_{max}}\ln\frac{\rho_{max}}{\rho} \quad (9)$$

For realistic purposes we can consider a density ρ_{min} for which the speed is v_{max}. Hence v_0 in (9) can be represented as $v_{max}/\ln(\frac{\rho_{max}}{\rho_{min}})$. Hence the logarithmic model (4) can be normalized as

$$\frac{v}{v_{max}} = \frac{1}{\ln\frac{\rho_{min}}{\rho_{max}}}\ln\frac{\rho}{\rho_{max}} \quad (10)$$

In the same way if we have a density ρ_{min} for which the speed is v_{max} on Equation (2), the v_f can be represented as $v_{max}/\ln(\frac{\rho_{max}}{\rho_{min}})$ and Equation (2) can also be transformed into (10). Equation (10) can be converted into the proposed generalized model (7) with proper choice of m and n. These values depend on the value of $\frac{\rho_{min}}{\rho_{max}}$ as shown in Table I.

Serial No	$\frac{\rho_{min}}{\rho_{max}}$	*m*	*n*
1	0.01	1.69	0.49
2	0.02	2.40	0.65
3	0.03	3.92	0.79
4	0.04	4.81	0.84
5	0.05	5.98	0.88
6	0.06	7.41	0.91
7	0.07	8.80	0.93
8	0.08	11.37	0.95
9	0.09	12.99	0.96
10	0.10	15.78	0.97

Table I: The approximate values of *m* and *n* for different value of $\frac{\rho_{min}}{\rho_{max}}$ used in the logarithmic model

C. Exponential Model

It is already known that $\rho \to 0$ implies $v \to v_{max}$ hence using this condition in (4), we get that $v_{max} = ev_0$. Hence the normalized of the exponential model (4) can be written as

$$\frac{v}{v_{max}} = \frac{1}{e}\exp\left(1 - \frac{\rho}{\rho_{max}}\frac{\rho_{max}}{\rho_0}\right)$$
$$= \exp\left(-\frac{\rho}{\rho_{max}}\frac{\rho_{max}}{\rho_0}\right) \quad (11)$$

Considering at density ρ_{max}, traffic speed is v_{min}, hence using (11) we can state that ρ_0 equals to $\rho_{max}/\ln(\frac{v_{max}}{v_{min}})$ hence exponential model can be normalized as

$$\frac{v}{v_{max}} = \exp\left(\frac{\rho}{\rho_{max}}\ln\frac{v_{min}}{v_{max}}\right) \quad (12)$$

Equation (12) can be converted into the proposed generalised model (7) with proper choice of m and n. These values depend on $\frac{v_{min}}{v_{max}}$ as shown in Table II.

Serial No	$\frac{v_{min}}{v_{max}}$	*m*	*n*
1	0.01	10.01	0.89
2	0.02	12.28	0.92
3	0.03	12.70	0.92
4	0.04	13.54	0.94
5	0.05	14.93	0.95
6	0.06	17.18	0.96
7	0.07	21.11	0.97
8	0.08	19.33	0.97
9	0.09	17.68	0.97
10	0.10	16.12	0.97

Table II: The approximate values of *m* and *n* for different values of $\frac{v_{min}}{v_{max}}$ used in exponential model

The proposed model can also be used for other speed-density functional forms which are derived or extended version of the linear, logarithmic, and/or exponential model(s). The values of *m* and *n* can be generated using a least square minimization method between the proposed functional form and any other functional forms.

V. EXTRACTING PARAMETER VALUES

In the proposed model (7), there are four unknown variables *m*, *n*, v_{max}, and ρ_{max} to get an approximate relation of traffic speed and density. From the real time traffic data we can approximate these values by measuring the speed at four different points of the density. By considering at instant where the speed is maximum, we can get v_{max}. Similarly considering the instant where speed is minimum (approximately equal to 0), we can get ρ_{max} . Now m and n must be computed; hence two measurements can be performed to know the speed where $\rho \approx \alpha\rho_{max}$ and to know the density where $v \approx \beta v_{max}$, where $0 < \alpha, \beta < 1$. Hence the value of m and n can be generated from the following approximate equation derived from (7) using simple algebraic manipulation

$$m \approx \frac{\left(\frac{v}{v_{max}}\right)_{\rho\approx\alpha\rho_{max}}^{} - (1-\alpha)}{\alpha} \quad (13)$$

$$n \approx \frac{\log\left\{(1-\beta)-(1-m)\left(\frac{\rho}{\rho_{max}}\right)_{v\approx\beta v_{max}}\right\}m^{-1}}{\log\left(\frac{\rho}{\rho_{max}}\right)_{v\approx\beta v_{max}}} \quad (14)$$

Due to the existence of a simple functional relationship, the parameters of the proposed model can be derived by using four simple approximate measurements. This procedure requires the information of v_{max} and ρ_{max}, which sometimes in real-time traffic measurement are not available.

In real-time data, sometimes ρ_{max} cannot be obtained directly, but the average maximum speed v_{max} can be approximated, since in real time traffic the traffic speed almost remains constant when traffic density is less and near to zero. Let ρ_1, ρ_2, ρ_3 be three measured traffic density points where $0<\rho_1<\rho_2<\rho_3<\rho_{max}$ and the speed or flow data are available for these traffic densities. Considering (ρ_1/ρ_{max}) is small such that $(\rho_1/\rho_{max})^n$ can be neglected, we can state that

$$(v/v_{max})_{\rho\approx\rho_1} \approx \quad 1-(1-m)(\rho_1/\rho_{max}) \qquad (15)$$

Since v_{max} can be approximated directly from the available traffic data, let define a variable α to simplify the calculation as follows,

$$\alpha = \frac{1-m}{\rho_{max}} = \frac{1-(v/v_{max})_{\rho\approx\rho_1}}{\rho_1} \qquad (16)$$

The value of α can be derived from the available data. Using (16), we can state that

$$m(\rho_i/\rho_{max})^n = 1-\alpha\rho_i - (v/v_{max})_{\rho\approx\rho_i} \quad \text{for } i = 2,3 \qquad (17)$$

Using simple algebraic manipulation the 'power factor' n can be approximated as

$$n \approx \frac{\ln[\{1-\alpha\rho_2-(v/v_{max})_{\rho\approx\rho_2}\}/\{1-\alpha\rho_3-(v/v_{max})_{\rho\approx\rho_3}\}]}{\ln(\rho_2/\rho_3)} \qquad (18)$$

The value of m and ρ_{max} can be approximated using following system of equations

$$\rho_{max} = (1-m)/\alpha \qquad (19)$$

$$\rho_{max} = \left[m/\{1-\alpha\rho_2 - (v/v_{max})_{\rho\approx\rho_2}\}\right]^{\frac{1}{n}}\rho_2 \qquad (20)$$

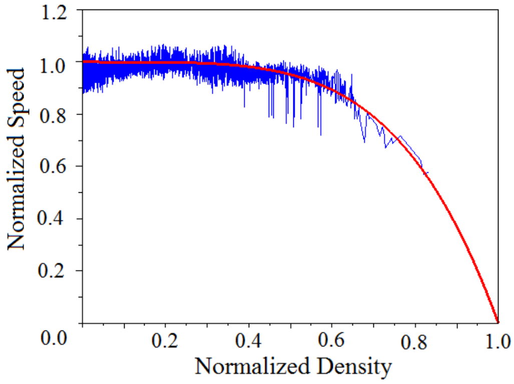
The solution of the last two equations can be found numerically or by graphical method.

Interestingly if $\alpha = 0$, the value of m and ρ_{max} can be estimated easily as $m = 1$ and

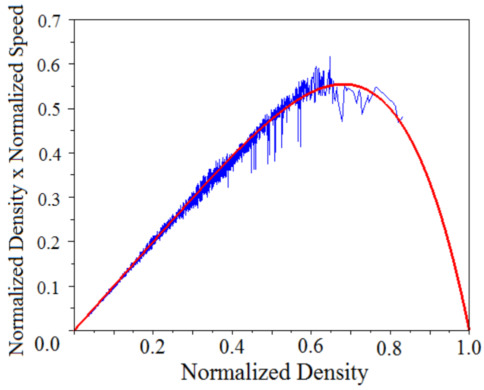
$$\rho_{max} \approx \left[1/\{1-(v/v_{max})_{\rho\approx\rho_2}\}\right]^{\frac{1}{n}}\rho_2 \qquad (21)$$

VI. Results & Discussion

To demonstrate how the proposed model can be used, we have taken the real time speed-flow data from [20]. From the data we sampled four points to estimate the various parameters (described in the previous section). The speed-density relationship is as follows,



(a)



(b)

Fig 2: (a) The normalized density-normalized speed and (b) a representation of normalized density vs traffic flow behavior. The blue curve shows the actual data derived from the real-time traffic data, the approximated model is represented by the red curve.

As shown in Figure 2(a), the proposed model is used to approximate the relationship between traffic density and the traffic speed, using this approximated relationship, the flow versus traffic density relationship is represented in Fig 2(b). The parameter extraction method used in the proposed functional form gives the values of parameters, m=1 and n = 4.39. These values can now be used to model vehicular traffic on that particular road.

VII. Conclusions

In this paper, we proposed a simple explicit model that relates the vehicular speed to the vehicular density on a given road segment. Our proposed model is of a general form and the parameters can be estimated with a very sparse set of data. We demonstrate how the parameters are to be estimated and did the same on a sample dataset. The estimated curves do follow the trends shown by the complete sample of data. Our model can thus be used easily within simulations of traffic on road segments. In this manner we believe we have made a reasonable contribution to the literature and towards the efforts of modelling vehicular traffic.

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